A Matrix Characterization for $\mathcal{MELL}$

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JELIA-98

October 12, 1998

Outline

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Motivation

Linear Logic (J.-Y. Girard 1987)

- Applications
  - Logic Programming
  - Planning
  - Meta-Logic
- Aspects
  - Resource sensitivity
  - Concurrency

Syntax

Def: Let Γ, Δ be multisets of formulas then $\Gamma \longrightarrow \Delta$ is a *sequent*.

Γ	input	program	current state + actions
Δ	output	query	goal state

Def: ($\mathcal{MELL}$-formulas)

	$\varphi^{\perp}$	negation
	$\mathbf{1}$	dummy resource
φ times ψ	$\varphi \otimes \psi$	conjunction
φ par ψ	$\varphi \wp \psi$	$\equiv (\varphi^{\perp} \otimes \varphi^{\perp})^{\perp}$
bang φ	$!\varphi$	reproduction
why not φ	$?\varphi$	$\equiv (!\varphi^{\perp})^{\perp}$

Additive connectives ($\&$, $\oplus$, $\top$, $\mathbf{0}$) are not considered in this talk.

Sequent Calculus Rules

$$\frac{}{\varphi \longrightarrow \varphi}\ ax \qquad \frac{\Gamma \longrightarrow \Delta}{\Gamma, \mathbf{1} \longrightarrow \Delta}\ \mathbf{1}{-}l \qquad \frac{}{\longrightarrow \mathbf{1}}\ \mathbf{1}{-}r$$

$$\frac{\Gamma \longrightarrow \Delta, \varphi}{\Gamma, \varphi^{\perp} \longrightarrow \Delta}\ {}^{\perp}{-}l \qquad \frac{\Gamma, \varphi \longrightarrow \Delta}{\Gamma \longrightarrow \Delta, \varphi^{\perp}}\ {}^{\perp}{-}r$$

$$\frac{\Gamma, \varphi, \psi \longrightarrow \Delta}{\Gamma, \varphi \otimes \psi \longrightarrow \Delta}\ \otimes{-}l \qquad \frac{\Gamma_1 \longrightarrow \Delta_1, \varphi \quad \Gamma_2 \longrightarrow \Delta_2, \psi}{\Gamma_1, \Gamma_2 \longrightarrow \Delta_1, \Delta_2, \varphi \otimes \psi}\ \otimes{-}r$$

$$\frac{\Gamma, \varphi \longrightarrow \Delta}{\Gamma, !\varphi \longrightarrow \Delta}\ !{-}l \qquad \frac{!\Gamma \longrightarrow ?\Delta, \varphi}{!\Gamma \longrightarrow ?\Delta, !\varphi}\ !{-}r$$

$$\frac{\Gamma, !\varphi, !\varphi \longrightarrow \Delta}{\Gamma, !\varphi \longrightarrow \Delta}\ c{-}! \qquad \frac{\Gamma \longrightarrow \Delta}{\Gamma, !\varphi \longrightarrow \Delta}\ w{-}!$$

Analytic Proof Search in Sequent Calculus

$$\frac{\vdots}{\longrightarrow A \otimes !A, ?(A^{\perp}), \perp}$$

Non-determinism

- choice of formula
- choice of rule

Goal of proof search: close all branches by axiom rules

Limitations:

- permutability of rules
- notational redundancies
- irrelevant reductions (lack of goal orientedness)

Goal: *search space representation* without these limitations

$\Longrightarrow$ matrix characterizations (Bibel '81, Andrews '81, Wallen '90)

Comparison of Basic Concepts

matrix characterization		sequent calculus
matrix	∼	sequent
path	∼	branch in derivation
(static creation)		(dynamic creation)
connection	∼	axiom
spanning set of connections	∼	all branches are closed
prefixes	∼	non-permutability of rules
(explicit)		(implicit)
⋮		⋮
complementary	∼	*valid*

Example Matrix

$\longrightarrow A \otimes !A, ?(A^{\perp}), \perp$

- polarity
- generic formula
- path
- prefix
- connection
- weakening map
- spanning, linear, relevant
- cardinality, unifiable

Weakening Map

Motivation: The structure of a sequent proof must allow to weaken formulas $\mathbf{1}^-$, $\bot^+$, $!\varphi^-$, and $?\varphi^+$.

Example:

$$\dfrac{\longrightarrow P, P^\bot, \bot \quad \longrightarrow \bot}{\longrightarrow P, P^\bot, \bot \otimes \bot} \otimes\text{-}r$$

The matrix has a spanning set of connections.

The weakening map contains .

Prefixes

- *Special positions* of four different types:
  - ϕ^M, ϕ^E : variable
  - ψ^M, ψ^E : constant
- A *prefix* is a string of special positions.
  (Construction is subtle but costs are linear.)
- Prefixes reflect the non-permutabilities of certain rules.
- A *prefix substitution* σ substitutes variable positions by strings of constant positions.
  - positions of type ϕ^M by strings from $(\psi^M)^*$
  - positions of type ϕ^E by strings from $(\psi^M \cup \psi^E)^*$

Prefix Unification

A prefix substitution σ is a unifier for $\mathcal{M}$, $\mathcal{C}$, and $\mathcal{W}$ if:

- The prefixes of all elements of all connections in $\mathcal{C}$ are equal under σ.
- The prefix of each element in $\mathcal{W}$ must fit to the prefix of some connection under σ.

Complementarity

Def: A matrix is *complementary* iff there is a set of connections $\mathcal{C}$, a weakening map $\mathcal{W}$, and a prefix substitution σ such that

- $\mathcal{C}$ *spans* $\mathcal{M}$
- $\langle \mathcal{C}, \mathcal{W} \rangle$ is *linear* for $\mathcal{M}$
- $\langle \mathcal{C}, \mathcal{W} \rangle$, has the *relevance property* for $\mathcal{M}$
- $\langle \mathcal{C}, \mathcal{W} \rangle$, has the *cardinality property* for $\mathcal{M}$
- σ is a unifier for $\mathcal{M}$, $\mathcal{C}$, and $\mathcal{W}$

Motivation of Requirements

requirement		property of corresponding sequent proof
spanning	$\sim$	axiom rules at leafs of a sequent proof
linear, relevance, cardinality	$\sim$	absence of structural rules
unifiability	$\sim$	non-permutabilities of rules

Characterization Theorem

A formula is valid in $\mathcal{MELL}$

if and only if

there is a multiplicity such that
the corresponding matrix is complementary.

Proof Sketch

- uniform notation for $\mathcal{MELL}$
- modification of Andreolis *dyadic* Σ_2' and *triadic* Σ_3' calculus
  - introduction of multiplicities
- construction of a matrix from a formula tree and a multiplicity
  - insertion of special positions
- position calculus Σ_{pos}
  - sequent proofs on matrices
- matrix characterization

$$\Sigma_1' \Longleftrightarrow \Sigma_2' \Longleftrightarrow \Sigma_3' \Longleftrightarrow \Sigma_{pos} \Longleftrightarrow \text{MC}$$

Conclusion

- Related Work:
  - other matrix characterizations: MLL (Fronhöfer '96), MLL/ALL (Galmiche '96), MLL (Mantel '96)
  - proof procedures: MLL (Kreitz Mantel Otten Schmitt '96)
  - matrix systems (Mantel Sandner '97)
- Open Tasks:
  - quantifiers with $\mathcal{MELL}$
  - extension of MLL proof procedure to $\mathcal{MELL}$
  - shorter prefixes are possible in some specific cases
  - induction in matrix based proof search